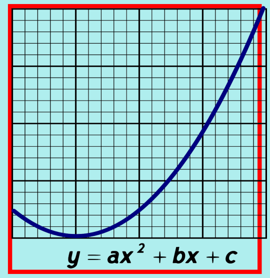
Math 25

Fall 2017

Lecture 8



Given: $\dfrac{(x+1)^2}{9} + \dfrac{(y-4)^2}{25} = 1$

① Ellipse

② Center $(h,k) = (-1,4)$

③ $a^2 = 25$, $b^2 = 9$, $a = 5$, $b = 3$

Major axis is vertical

Since a is under y

④ $c^2 = a^2 - b^2$ $\quad c^2 = 16$, $\boxed{c = 4}$

⑤ Eccentricity $\quad e = \dfrac{c}{a} \quad e = \dfrac{4}{5}$

⑤ Graph

⑥ foci $(-1, 4 \pm 4)$

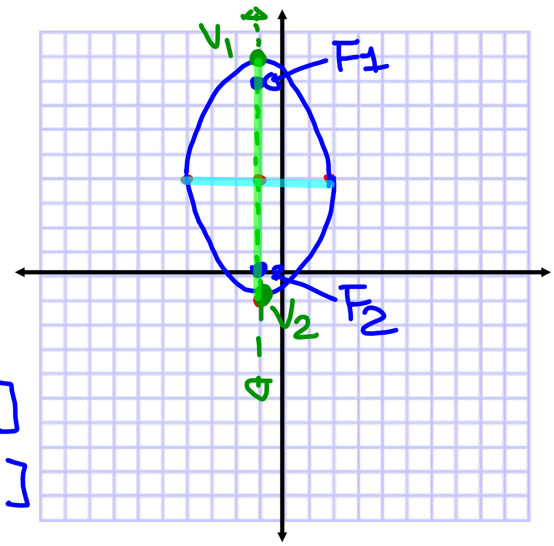
c

h

k

⑦ Domain $[-4, 2]$

Range $[-1, 9]$



Consider

$$5x^2 + 8y^2 - 40x + 16y + 48 = 0$$

① Ellipse, Not equal, both have Same Sign

② Write it in standard form

$$5x^2 - 40x \qquad + 8y^2 + 16y \qquad = -48$$

$$5(x^2 - 8x + 16) + 8(y^2 + 2y + 1) = -48 + 80 + 8$$

$$5(x-4)^2 + 8(y+1)^2 = 40$$

Divide by 40 to make RHS 1,

$$\frac{(x-4)^2}{8} + \frac{(y+1)^2}{5} = 1$$

Center $(4,-1)$

$a^2=8$ $\quad b^2=5$, major axis horizontal because a is under x.

$a=\sqrt{8}$ $\quad b=\sqrt{5}$

$a=2\sqrt{2}$

$c^2=a^2-b^2 \quad c^2=8-5 \quad c^2=3 \quad c=\sqrt{3}$

$e=\frac{c}{a} \quad e=\frac{\sqrt{3}}{\sqrt{8}} \quad e=\frac{\sqrt{3}\sqrt{8}}{\sqrt{8}\sqrt{8}} \quad e=\frac{\sqrt{24}}{8}=\frac{\cancel{2}^{1}\cancel{\sqrt{4}}\sqrt{6}}{\cancel{8}_{4}}$

$e=\frac{\sqrt{6}}{4}$

$$\frac{(x-4)^2}{8} + \frac{(y+1)^2}{5} = 1$$

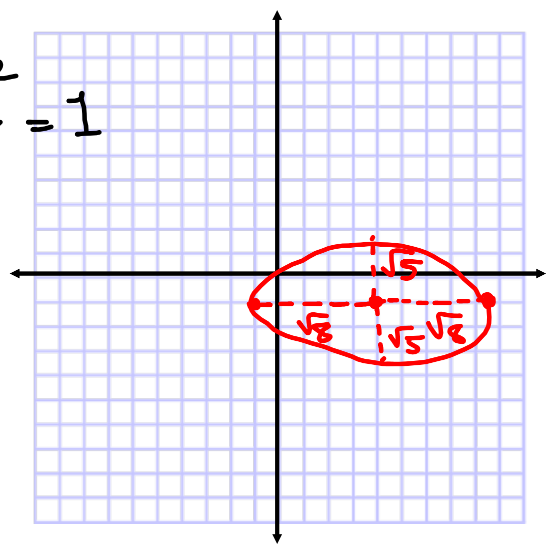
Vertices

$(4 \pm \sqrt{8}, -1)$

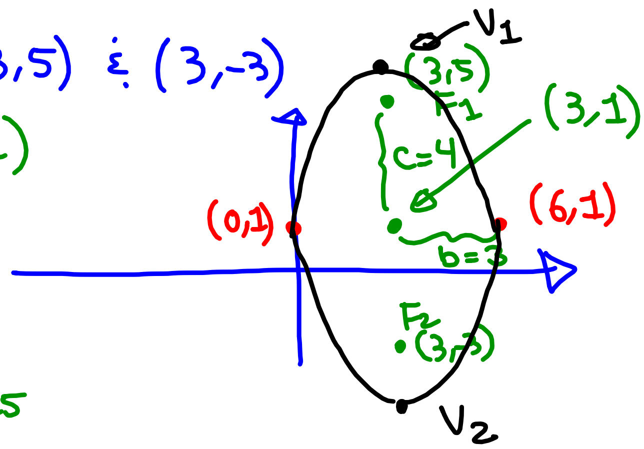
$= (4 \pm 2\sqrt{2}, -1)$ a

Foci $\quad$ c

$(4 \pm \sqrt{3}, -1)$



Write / Draw an ellipse with endpoints on minor axis $(0,1)$ & $(6,1)$ with foci $(3,5)$ & $(3,-3)$



Center $(h,k)=(3,1)$

$c^2=a^2-b^2$

$4^2=a^2-3^2$

$16+9=a^2 \qquad a^2=25$

$a=5$

$$\frac{(x-3)^2}{9}+\frac{(y-1)^2}{25}=1 \qquad e=\frac{c}{a}=\frac{4}{5}$$

$e=4/5$

Consider $\dfrac{(x+3)^2}{16}-\dfrac{(y-2)^2}{9}=1$

Hyperbola

Horizontal hyperbola because x is first.

Major axis is horizontal, a is under x.

Center $(-3,2)$

$a^2=16 \qquad b^2=9 \qquad c^2=a^2+b^2 \qquad c^2=25$

$a=4, \; b=3, \; c=5 \qquad e=\dfrac{c}{a}=\dfrac{5}{4}$

Verices

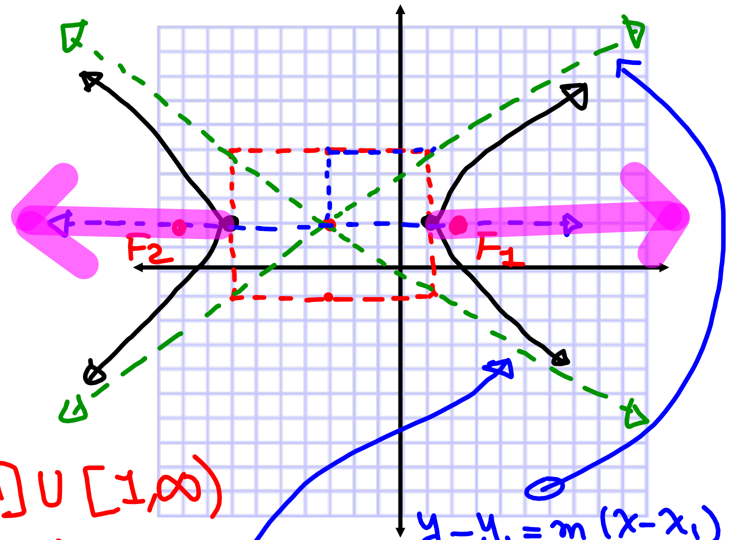
$(-3 \pm 4, 2)$

a

Foci

$(-3 \pm 5, 2)$

c



Domain: $(-\infty, -7] \cup [1, \infty)$

Range: $(-\infty, \infty)$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{-3}{4}(x+3)$ $\quad$ $y - 2 = \frac{3}{4}(x - -3)$

$y - 2 = \frac{3}{4}(x+3)$

Consider

$$7y^2 - 11x^2 - 44x + 56y - 145 = 0$$

① Hyperbola, have opposite signs.

② $7y^2 + 56y \quad -11x^2 - 44x \quad = 145$

$$7(y^2 + 8y + 16) - 11(x^2 + 4x + 4) = 145 + 7(16) - 11(4)$$

$$7(y+4)^2 - 11(x+2)^2 = 213$$

Divide by 213 to make RHS 1

$$\frac{(y+4)^2}{\frac{213}{7}} - \frac{(x+2)^2}{\frac{213}{11}} = 1$$

$$\frac{213}{7} > \frac{213}{11} \qquad a^2 = \frac{213}{7}, \quad b^2 = \frac{213}{11}$$

$$c^2 = a^2 + b^2 \qquad c^2 = \frac{213}{7} + \frac{213}{11} = \frac{213(18)}{77}$$

Center $(-2, 4)$

Vertical hyperbola

Consider $-9x^2 + 16y^2 + 64y - 512 = 0$

Hyperbola — opposite signs

$$16y^2 + 64y \qquad -9x^2 \qquad = 512$$

$$16(y^2 + 4y + 4) - 9x^2 = 512 + 16 \cdot 4$$

$$16(y+2)^2 - 9x^2 = 576$$

Make RHS 1, Simplify

$$\frac{(y+2)^2}{36} - \frac{x^2}{64} = 1$$

Center $(0, -2)$

Vertical Hyperbola

$a^2 = 36 \quad a = 6$

$b^2 = 64 \quad b = 8$

$c^2 = a^2 + b^2 \qquad c = 10$

Vertices

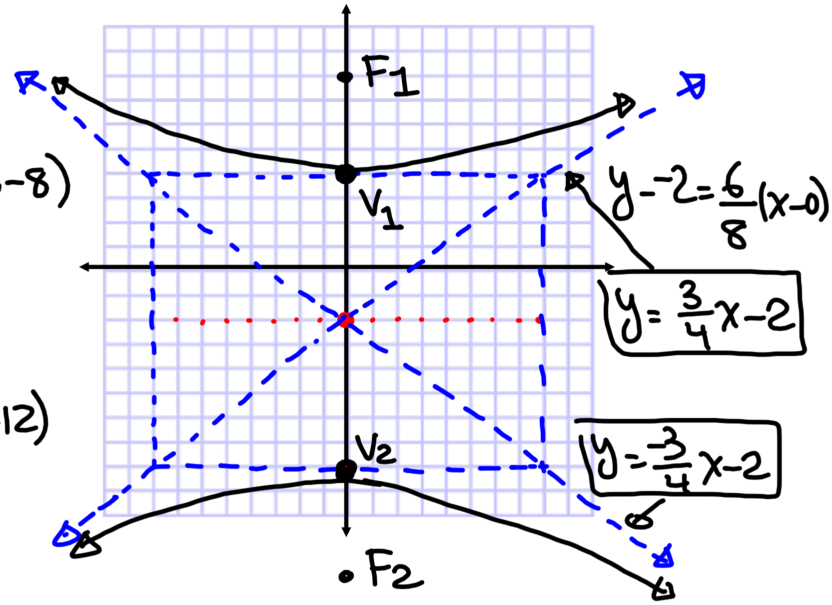
$(0, -2 \pm 6)$

$\Rightarrow (0, 4), (0, -8)$

Foci

$(0, -2 \pm 10)$

$= (0, 8), (0, -12)$

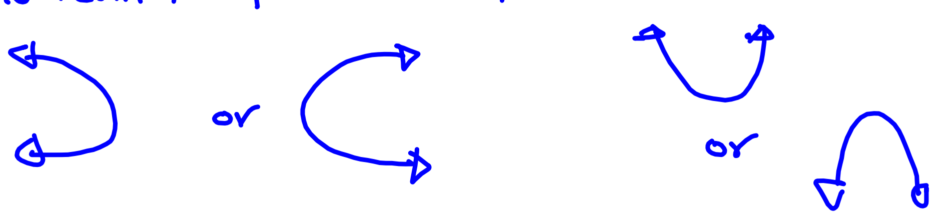


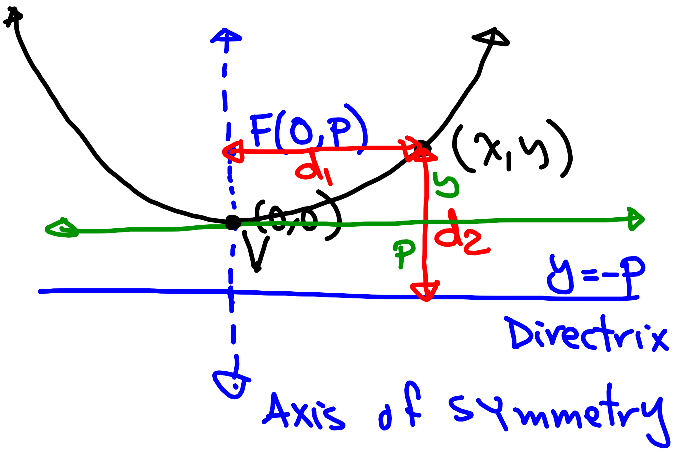
D: $(-\infty, \infty)$

R: $(-\infty, -8] \cup [4, \infty)$

Parabola is the set of all points (x, y) that are same distance from a fixed point, Focus, and a fixed line, directrix.

horizontal parabola , vertical parabola





Parabola never crosses the directrix.

Parabola is symmetric with respect to axis of sym.

$d_1 = d_2$

$$\sqrt{(x-0)^2+(y-p)^2} = y + p$$

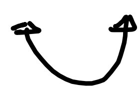
Square both sides $\Rightarrow$ $(x-0)^2+(y-p)^2=(y+p)^2$

$$x^2 + (y-p)(y-p) = (y+p)(y+p)$$

$$x^2 + \cancel{y^2} - 2yp + \cancel{p^2} = \cancel{y^2} + 2yp + \cancel{p^2}$$

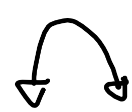
$$x^2 = 4py$$

If $p>0$



Vertex $(0,0)$

If $p<0$



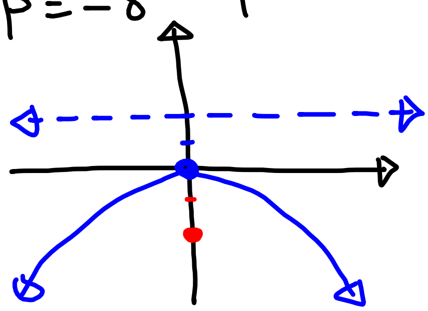
Draw $x^2 = -8y$ $\Rightarrow 4p = -8$ $\quad p = -2$

$x^2 = 4py$

Focus $(0,p)$ $(0,-2)$

Directrix $y=-p$ $\quad y=2$

Vertex $(0,0)$



Consider $\frac{1}{2}(x-3)^2 = 2y + 4$

we know $x^2 = 4py$

$V(0,0)$

Multiply by 2

$(x-3)^2 = 2(2y+4)$

$(x-3)^2 = 4y + 8$

$x-3=0$ $\quad$ $x=3$

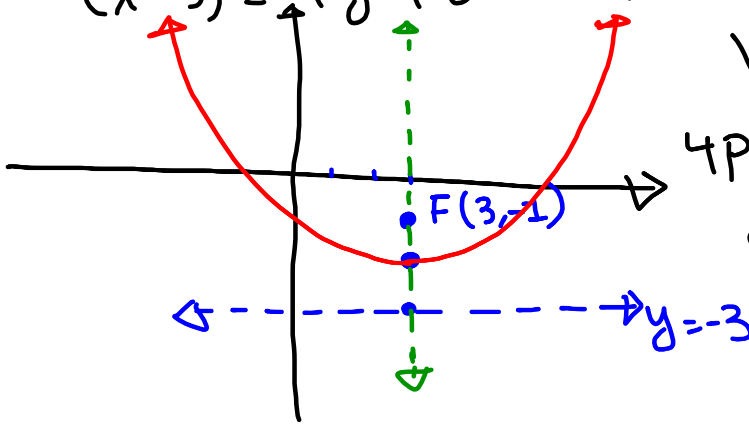
$y+2=0$ $\quad$ $y=-2$

$\rightarrow (x-3)^2 = 4(y+2)$

$V(3,-2)$

$4p = 4$ $\quad$ $p=1$

opens upward



Consider $\frac{1}{16}(x+3)^2 = \frac{1}{2}y - 2$

$(x-h)^2 = 4p(y-k)$

Multiply by 16

$(x+3)^2 = 8y - 32$

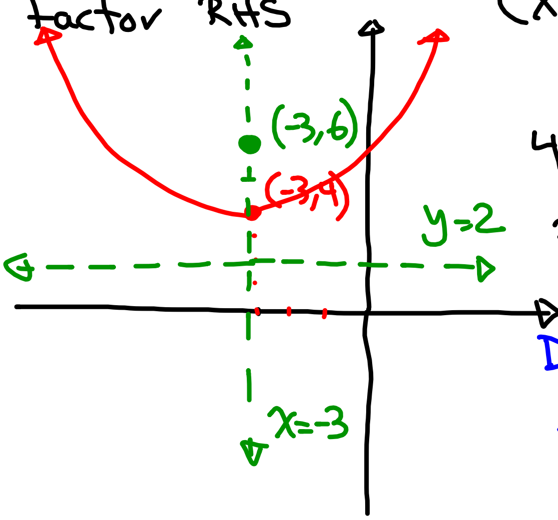
Factor RHS

$(x+3)^2 = 8(y-4)$

vertex $(-3,4)$

$4p=8$ $\quad$ $P=2$

$P>0 \rightarrow$ open upward



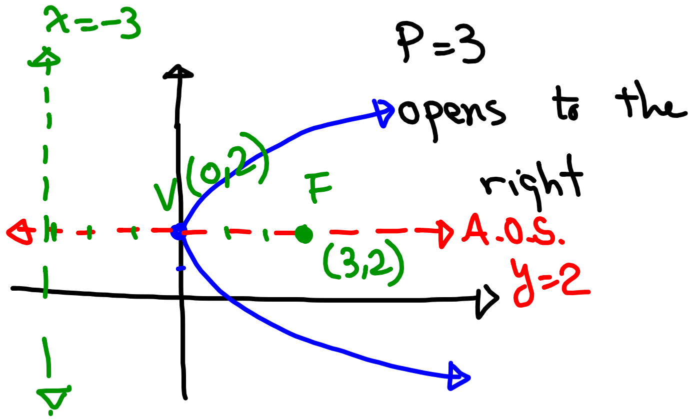
Domain: $(-\infty, \infty)$

Range: $[4, \infty)$

Consider $(y-2)^2 = 12x$

Vertex $(0,2)$ $\quad 4p = 12$

$p = 3$



Now think of

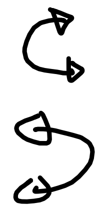
$y^2 = 4px$

horizontal parabola

Vertex $(0,0)$

$p>0$

$p<0$



Consider $4y^2 - 12y - 12x + 21 = 0$

when one of variables is squared → Parabola.

$(x-h)^2 = 4p(y-k)$

or

$(y-k)^2 = 4p(x-h)$

$4y^2 - 12y = 12x - 21$

$4\left(y^2 - 3y + \frac{9}{4}\right) = 12x - 21 + 4 \cdot \frac{9}{4}$

$4\left(y - \frac{3}{2}\right)^2 = 12x - 21 + 9$

$4\left(y - \frac{3}{2}\right)^2 = 12x - 12$

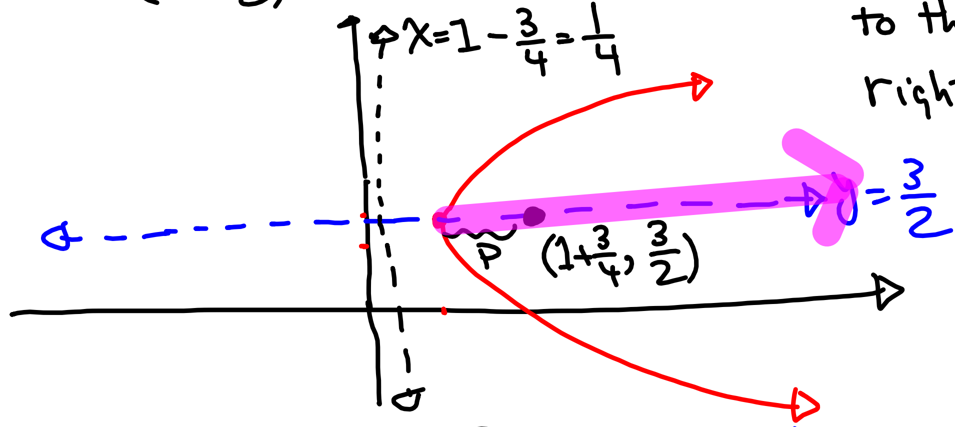
Divide by 4 → $\left(y - \frac{3}{2}\right)^2 = 3x - 3$

$\left(y - \frac{3}{2}\right)^2 = 3(x-1)$

$(y-\frac{3}{2})^2=3(x-1)$

Horizontal parabola

Center $(1, \frac{3}{2})$, $4p=3$, $p=\frac{3}{4}>0$ opens to the right



Domain: $[1,\infty)$ Range: $(-\infty,\infty)$

Consider $16y^2-56y-16x+81=0$

① Parabola, because only one variable is squared.

$(x-h)^2=4p(y-k)$

② Stand. form

$(y-k)^2=4p(x-h)$

$16y^2-56y \qquad =16x-81$

$16(y^2-\frac{56}{16}y \qquad)=16x-81$ $\quad \frac{1}{2}\cdot\frac{7}{2}=\frac{7}{4}$

$16(y^2-\frac{7}{2}y+\frac{49}{16})=16x-81+16\cdot\frac{49}{16}$

$16(y-\frac{7}{4})^2=16x-81+49$

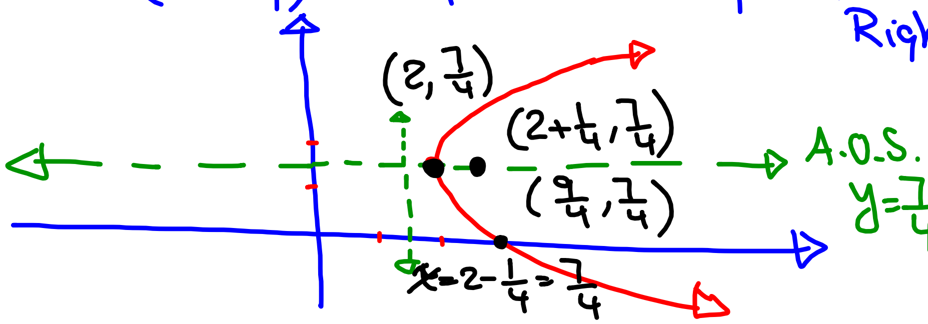
$16\left(y-\frac{7}{4}\right)^2 = 16x - 32$

Make to become 1 $\longrightarrow$ $(y-k)^2 = 4p(x-h)$

Divide by 16

$$\left(y-\frac{7}{4}\right)^2 = x-2 \Rightarrow \left(y-\frac{7}{4}\right)^2 = 1(x-2)$$

Vertex $\left(2, \frac{7}{4}\right)$ $\quad 4p = 1 \quad p = \frac{1}{4}$ $\quad$ opens Right



$y = ax^2 + bx + c \qquad\qquad x = ay^2 + by + c$

$V\left(\frac{-b}{2a}, \text{---}\right) \qquad\qquad \left(\quad , \frac{-b}{2a}\right)$

a tells up which direction opens

$0 < a < 1 \rightarrow$ opens wide/ Narrow

$a > 1$

General Equation of a Conic Section

$Ax^2 + Cy^2 + Dx + Ey + F = 0$

1) $A = C \Rightarrow$ Circle

2) $AC > 0 \Rightarrow$ Ellipse

3) $AC < 0 \Rightarrow$ Hyperbola

4) $AC = 0 \Rightarrow$ Parabola ($A=0$ or $C=0$)

$x^2 + y^2 + 2x - 12y + 21 = 0$

$A=1 \quad C=1 \quad D=2 \quad E=-12 \quad C=21$

$A=C \rightarrow$ Circle

$x^2 + 2x + 1 + y^2 - 12y + 36 = -21 + 1 + 36$

$(x+1)^2 + (y-6)^2 = 16$

Center $(-1, 6)$ Radius 4.

$16x^2 + y^2 + 32x = 0$

$A=16 \quad C=1 \quad D=32 \quad E=0 \quad F=0$

$A \neq C \rightarrow$ Not a Circle

$AC > 0 \rightarrow$ Ellipse

$16x^2 + 32x \quad + y^2 = 0$

$16(x^2 + 2x + 1) + y^2 = 0 + 16$

$16(x+1)^2 + (y-0)^2 = 16 \quad \frac{(x+1)^2}{1} + \frac{(y-0)^2}{16} = 1$

$x^2 - 2x - 10y - 29 = 0$

$A=1, \ C=0, \ D=-2, \ E=-10, \ F=-29$

$AC=0 \rightarrow$ Parabola

$x^2 - 2x + 1 = 10y + 29 + 1$

$(x-1)^2 = 10y + 30 \quad (x-1)^2 = 10(y+3)$

$$-144x^2+25y^2-1152x-50y-5879=0$$

$A=-144$ $C=25$ $D=-1152$ $E=-50$ $F=-5879$

$AC<0 \rightarrow$ Hyperbola

$$-144x^2-1152x \quad +25y^2-50y \quad =5879$$

$$-144(x^2+8x+16)+25(y^2-2y+1)=5879$$

$-144(16)$

$+25(1)$

$$-144(x+4)^2+25(y-1)^2=3600$$

Make RHS become 1

$$\frac{-144(x+4)^2}{3600}+\frac{25(y-1)^2}{3600}=1$$

$$-\frac{(x+4)^2}{25}+\frac{(y-1)^2}{144}=1$$

$$\frac{(y-1)^2}{144}-\frac{(x+4)^2}{25}=1$$

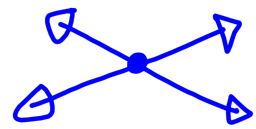
$$9x^2 - 4y^2 - 90x - 8y + 221 = 0$$

$$9x^2 - 90x - 4y^2 - 8y + 221 = 0$$

$$9(x^2 - 10x + 25) - 4(y^2 + 2y + 1) = -221$$

$$9 \cdot 25 - 4 \cdot 1$$

$$9(x-5)^2 - 4(y+1)^2 = 0$$



Degenerate equation

$$9(x-5)^2 = 4(y+1)^2 \Rightarrow (y+1)^2 = \frac{9}{4}(x-5)^2$$

$$y + 1 = \pm \frac{3}{2}(x-5)$$

$$x^2 + y^2 + 12x + 37 = 0$$

$$x^2 + 12x + 36 + y^2 = -37 + 36$$

$$(x+6)^2 + y^2 = -1$$

Not such equation makes Sense.

Exam II Next week

Review exam 1

Review weekly Quizzes

Inverse matrix, Cramer's rule,
Conic Sections.

weekly QZ will be posted,
Due next week.

Looking ahead:

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

n is # of terms

$n=1 \to$ LHS $=1$
$\to$ RHS $=1^2$ $\Rightarrow 1=1$

$n=2 \to$ LHS $= 1+3$
RHS $= 2^2$ $\Rightarrow 4=4$

$n=3$
LHS $\to$
$1+3+5=9$
RHS $\to 3^2=9$

$n=4$ LHS $= 1+3+5+7 = 16$ ✓
RHS $= 4^2 = 16$ ✓

I verified that it works for $n=1$, $n=2$, $n=3$, and $n=4$.

If $n=10$

LHS $= 1+3+5+7+9+11+13+15+17+19$

RHS $= 10^2$ $\qquad 100 = 100$ ✓

To prove that this works for all n, we use Mathematical Induction.

Assume that it works for $n=k$

$$1+3+5+7+\dots+(2k-1) = k^2$$

Add the next term, based on the pattern to both sides.

$$\underbrace{1+3+5+7+\dots+(2k-1)+[2k-1+2]}_{} = k^2+[2k-1+2]$$

$$= k^2+2k+1$$

How many terms?

$k+1$ terms

$$= (k+1)^2$$

$$2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$$

$n=1$

LHS $= 2^0 = 1$ ✓

RHS $= 2^1 - 1 = 2-1 = 1$

$n=2$ LHS $= 2^0 + 2^1 = 1+2 = 3$ ✓

RHS $= 2^2 - 1 = 4 - 1 = 3$

$n=3$ LHS $= 2^0 + 2^1 + 2^2 = 1+2+4 = 7$ ✓

RHS $= 2^3 - 1 = 8 - 1 = 7$

Assume it works for $n=k$

$$2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1$$

Add the next term, based on the pattern to both sides

$$\underbrace{2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{k-1} + 2^{k-1+1}}_{} = 2^k - 1 + 2^{k-1+1}$$

How many terms?

$k+1$ terms

$= 2^k - 1 + 2^k$

$= 2 \cdot 2^k - 1 = 2^{k+1} - 1$

$$1 + 2 + 3 + 4 + 5 + \cdots + n = \frac{n(n+1)}{2}$$

$n = 1$ LHS $= 1$

RHS $= \frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1$ ✓

$n = 2$ LHS $= 1 + 2 = 3$

RHS $= \frac{2(2+1)}{2} = 3$ ✓

$n = 3$ LHS $= 1 + 2 + 3 = 6$

RHS $= \frac{3(3+1)}{2} = \frac{3 \cdot 4}{2} = \frac{12}{2} = 6$ ✓

Assume it works for $n = k$

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Add the next term, based on the pattern to both sides

$$\underbrace{1 + 2 + 3 + \cdots + k + k+1}_{(k+1)\text{ terms}} = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+1+1)}{2} = \frac{(\#\text{ of terms})(\#\text{ of terms} + 1)}{2}$$